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Squares of Automorphic Forms on Quaternion Algebras and Central Values of L-Functions of Modular Forms

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INTRODUCTION

Let f be a modular form of weight 2, trivial character and prime level p that is an eigenform of all Hecke operators. By the correspondence of Eichler, Shimizu, and Jacquet–Langlands there is an associated Hecke eigenform φ with the same Hecke eigenvalues on the adelic quaternion algebra $D_{\mathbf{A}}^{\times}$ over $\mathbf{Q}$ which is ramified at ∞ and p; the form φ is right invariant under the adelic units of a maximal order R of D. As is well known this quaternionic form φ is determined by the values at a set of representatives $y_i \in D_{\mathbf{A}}^{\times}$ of the ideal classes I_i of the maximal order R (i.e., $I_i = R y_i^{-1}$ or, equivalently, $D_{\mathbf{A}}^{\times} = \bigcup_{i=1}^{h} D_{\mathbf{Q}}^{\times} y_i R_{\mathbf{A}}^{\times}$). The form φ can also be identified with a function on the set of supersingular elliptic curves modulo p.

In this latter setting a curious connection between the central critical value of the L-functions of f and of $\mathrm{Sym}^3(f)$ and the moments of φ has been observed empirically by Zagier (using computer calculations of Mestre). Let e_i be the number of units of the right order R_i of I_i and put $M_r = \sum_i (\varphi(y_i)^r / e_i)$. Then the third moment M_3 gives up to a period factor the product of $L(f, 1)$ and a square root of $L(\mathrm{Sym}^3(f), 2)$. Moreover, in all cases studied the values of φ appear in pairs of opposite sign (and hence

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the moments M_r for odd r are all zero) if (and only if) either one of $L(f, 1)$, $L(\mathrm{Sym}^3(f), 2)$ is zero. The latter condition is equivalent to the vanishing of the central critical value $L(f\otimes f\otimes f, 2)$ of the triple product L-function of $(f\otimes f\otimes f)$ or, in view of the observed proportionality of M_3 and $L(f, 1)$, to the vanishing of M_3.

This phenomenon can be explained quite naturally if the vanishing of $L(f\otimes f\otimes f, 2)$ is caused by a minus sign in the functional equation of the triple product L-function: Similar to the Fricke involution w_p on the space of modular forms of level p one has a natural involution $\tilde{w}_p$ on the space of automorphic forms on the adelic quaternion algebra. From the results of [9, 4] it is easy to deduce that in this case φ is an eigenfunction with eigenvalue -1 of the permutation of $\tilde{w}_p$. In the case of a $+$ sign in the functional equation, however, there is no involution known that could be responsible for the observed behaviour of φ.

The identity for M_3 has been proven meanwhile by Gross and Kudla in [9] as a consequence of their work on the triple product L-function and extended to higher weights by us in [4]. The observation concerning the vanishing of the higher odd moments remained unproven. In fact, an extended computation of examples performed by F. Wichelhaus in Saarbrücken produced two cases ($p=2089$, $p=6199$) where the third moment is the last one that vanishes (details will appear elsewhere). Instead Gross and Kudla could prove that the vanishing of $L(f, 1)$ implies that φ is orthogonal to all squares of eigenfunctions ψ of the relevant type on $D_{\mathbf{A}}^{\times}$, i.e.,

$$\sum_i \frac{\varphi(y_i)(\psi(y_i))^2}{e_i}=0, \tag{1}$$

using again their results on the triple product L-function and the fact that L-functions of type

$$L(f\otimes \mathrm{Sym}^2(g))$$

are known to be entire [12]. Statement (1), which had been noticed empirically by Birch [1], has some similarity with Zagier's observation. Both are implied by the involution argument from above in the case of a minus sign in the functional equation of $L(f\otimes f\otimes f)$. From our work [4] on the central critical value of triple L-functions for modular forms of arbitrary (even) weights $k_1\leqslant k_2\leqslant k_3$ satisfying

$$k_1+k_2>k_3,$$

it is more or less clear that a statement similar to (1) can be deduced in the same way for f of higher weight k.

In the present note we take a different approach using our results on the (non-)vanishing of Yoshida liftings to prove a version (see Theorem) of (1) for higher weights. The advantage of this method is that it allows us in addition to strengthen the result and to prove a kind of converse to it. We can thus characterize the vanishing of $L(f, k/2)$ completely in terms of elementary properties of the associated automorphic form on the adelic quaternion algebra.

A rather surprising application of our results connecting the nonvanishing of $L(f, k/2)$ with the nonvanishing of a special value of $L(f \otimes \mathrm{Sym}^2(g))$ for suitable g is given in a corollary to the theorem.

RESULTS

We review the relevant setup briefly and refer to [5] for details and references. Let D be a definite quaternion algebra over $\mathbf{Q}$ and R an Eichler order of square free level N in D. We will usually decompose N as $N = N_1 N_2$ where N_1 is the product of the primes dividing N that are ramified in D. On D we have the involution $x \mapsto \bar{x}$, the (reduced) trace $\mathrm{tr}(x) = x + \bar{x}$ and the (reduced) norm $n(x) = x\bar{x}$. We write $Im(x) = (x - \bar{x})$. The special orthogonal group H^+ of the quadratic form $q(x) = n(x)$ on D is isomorphic to $\{(x_1, x_2) \in D^\times \times D^\times \mid n(x_1) = n(x_2)\}/Z(D^\times)$ (as algebraic group).

For $\nu \in \mathbf{N}$ let $U_\nu^{(0)}$ be the space of homogeneous harmonic polynomials of degree ν on $\mathbf{R}^3$ and view $P \in U_\nu^{(0)}$ as a polynomial on $D_\infty{}^{(0)} = \{x \in D_\infty \mid \mathrm{tr}(x) = 0\}$ by putting $P(\sum_{i=1}^3 x_i e_i) = P(x_1, x_2, x_3)$ for an orthonormal basis $\{e_i\}$ of $D_\infty{}^{(0)}$ with respect to the norm form n. In the same way we fix an orthonormal basis of D_∞ extending the one from above and use it to identify (harmonic) polynomials in 4 variables with (harmonic) polynomial functions on D_∞.

The representation of $D_\infty^\times$ on $U_\nu^{(0)}$ by conjugation of the argument is denoted by τ_ν. By $\langle\!\langle\ ,\ \rangle\!\rangle$ we denote the invariant scalar product in the representation space $U_\nu{}^{(0)}$ (where the choice of normalization is irrelevant for our purposes). It is a consequence of a result of Littelmann ([11], p. 145) that for $\nu_1 \leqslant \nu_2 \leqslant \nu_3$ satisfying $\nu_1 + \nu_2 \geqslant \nu_3$ there is a unique (up to scalars) nontrivial $D_\infty^\times$-invariant trilinear form T on the product $U_{\nu_1}^{(0)} \times U_{\nu_2}^{(0)} \times U_{\nu_3}^{(0)}$. The $H^+(\mathbf{R})$-space $U_\nu^{(0)} \otimes U_\nu^{(0)}$ is isomorphic to the $H^+(\mathbf{R})$-space $U_{2\nu}$ of harmonic polynomials on D_∞ of degree 2ν. In particular for integers μ, ν with $2\nu \geqslant \mu$ the trilinear form T gives rise to a bilinear map (also denoted by T) from $U_\mu{}^{(0)} \times U_{2\nu}$ to $\mathbf{C}$ which is invariant under the action of $D_\infty^\times$ (acting on the second factor by the diagonal embedding of $D_\infty^\times$ into $D_\infty^\times \times D_\infty^\times$). $T(P_1, P_2 \otimes P_3)$ can then be obtained by taking the scalar product of $P_1 \in U_\mu{}^{(0)}$ with the projection of $P_2 \otimes P_3$ onto the unique component of $U_\nu^{(0)} \otimes U_\nu^{(0)}$ that is isomorphic to $U_\mu^{(0)}$. The existence of a unique trilinear form T is equivalent to the existence of a unique (up to scalars) non-trivial intertwining map from $U_\mu{}^{(0)}$ to $U_{2\nu}$ (both spaces viewed as $D_\infty^\times$-spaces).

We choose a double coset decomposition

$$D_{\mathbf{A}}^\times = \bigcup_{i=1}^{h=h(M_1, M_2)} D_{\mathbf{Q}}^\times y_i R_{\mathbf{A}}^\times$$

of the adelic multiplicative group of D with $R_{\mathbf{A}}^\times = D_\infty^\times \times \prod_{p \neq \infty} R_p^\times$ and representatives y_i with $n(y_i) = 1$ and $(y_i)_\infty = 1$. By $\mathscr{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$ we denote the space of functions $\varphi\colon D_{\mathbf{A}}^\times \to U_\nu^{(0)}$ (called functions of weight ν in the sequel) satisfying $\varphi(\gamma x u) = \tau_\nu(u_\infty^{-1})\, \varphi(x)$ for $\gamma \in D_{\mathbf{Q}}^\times$ and $u = u_\infty u_f \in R_{\mathbf{A}}^\times$, where $R_{\mathbf{A}}^\times = D_\infty^\times \times \prod_p R_p^\times$ is the adelic group of units of R. The natural inner product on this space is given by

$$\langle \varphi, \psi \rangle = \sum_{i=1}^{h} (\langle\!\langle \varphi(y_i), \psi(y_i) \rangle\!\rangle / e_i).$$

Following [10] we denote by $\mathscr{A}_{\mathrm{ess}}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$ the essential part of $\mathscr{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$. It consists of the functions φ that are orthogonal to all $\rho \in \mathscr{A}(D_{\mathbf{A}}^\times, (R'_{\mathbf{A}})^\times, \tau)$ for orders $R' \supseteq R$ strictly containing R.

The correspondence of Eichler, Shimizu, and Jacquet–Langlands matches newforms (Hecke eigenforms) in the space of cusp forms $S^{2+2\nu}(\Gamma_0(N))$ of weight $2+2\nu$ and level N on the modular forms side with Hecke eigenforms with the same eigenvalues in $\mathscr{A}_{\mathrm{ess}}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \nu)$ on the adelic side. It is compatible with involutions in the following sense: For each $p \mid N$ we have an involution $\widetilde{w_p}$ of $\mathscr{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \tau)$. If φ corresponds to $f \in S^{2+2\nu}(\Gamma_0(N))$ then the eigenvalue of f under the Atkin-Lehner involution w_p is equal to that of φ under $\tilde{w}_p$ if D splits at p and equal to minus that of φ under $\tilde{w}_p$ if D_p is a skew field.

We fix now some $f \in S^{2+2\nu}(\Gamma_0(N))$. If f has eigenvalue -1 under the Fricke involution $w_N = \prod_{p \mid N} w_p$ it is possible to choose the decomposition $N = N_1 N_2$ in such a way that the $\widetilde{w_p}-$ eigenvalues of φ are all $+1$.

Our main result is then the following theorem (using the notations established above):

THEOREM. *Let f be a newform of weight $k = 2 + 2\mu$ and trivial character with respect to the group $\Gamma_0(N)$ for squarefree N and assume that $f \mid w_N = -f$. Let N be decomposed as $N_1 N_2$ as above with $p \mid N_1$ if and only if $f \mid w_p = -f$. Let D be the quaternion algebra over $\mathbf{Q}$ ramified at ∞ and the primes dividing N_1 and let R be an Eichler order of level N in D. Let $\varphi \in \mathscr{A}(D_{\mathbf{A}}^\times, R_{\mathbf{A}}^\times, \mu)$ correspond to f under Eichler's correspondence.*

Then $L(f, k/2) = 0$ if and only if one has

$$\sum_{i=1}^{h} \frac{T(\varphi(y_i), \psi(y_i) \otimes \psi(y_i))}{e_i} = 0 \tag{2}$$

for all Hecke eigenforms $\psi \in \mathscr{A}_{\text{ess}}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times}, v)$ with $2v \geqslant \mu$; this condition is then satisfied for all Hecke eigenforms $\psi \in \mathscr{A}(D_{\mathbf{A}}^{\times}, R_{\mathbf{A}}^{\times}, v)$. For even μ, condition (2) is already satisfied for all v with $2v \geqslant \mu$ if it is true for $2v = \mu$ and for $2v = \mu + 2$.

By analogy with the case $v = \mu = 0$ we also say that $L(f, k/2) = 0$ if and only if φ is orthogonal to all squares of (essential) Hecke eigenforms of level N on the adelic quaternion algebra. If $4 \mid k$ (or equivalently if μ is odd) then $L(f, k/2) = 0$ for all f with $f \mid w_N = -f$; our result then says that all φ corresponding to newforms f as above are orthogonal to all squares of (essential) eigenforms.

We proved in [4]) that the square of the left hand side of (4) is (up to an explicitly known nonzero factor) equal to $L(f \otimes g \otimes g, (k/2) + k_g - 1)$, where g is the normalized newform of weight $k_g = 2 + 2v$ corresponding to ψ. Using this, we can reformulate the theorem in terms of modular forms: $L(f, k/2) \neq 0$ is equivalent to the existence of some newform g of weight $k/2 + 1$ or of weight $k/2 + 3$ such that the value of $L(f \otimes g \otimes g, s)$ at the central critical point $(k/2) + k_g - 1$ is not zero.

The "only if"-part of our theorem, stating that $L(f, k/2) = 0$ implies the vanishing of all the $L(f \otimes g \otimes g, (k/2) + k_g - 1)$, can also be deduced in an alternative way: In view of the factorization

$$L(f \otimes g \otimes g, s) = L(f, s - k_g + 1)\, L(f \otimes \operatorname{Sym}^2(g), s)$$

(as in [9] for $k = k_g = 2$) it follows from the entireness of $L(f \otimes \operatorname{Sym}^2(g), s)$, see [12].

We can also reformulate the "if"-part of the theorem in a slightly different way in terms of modular forms (using the facts mentioned above) and thus obtain from the theorem the following somewhat surprising result.

COROLLARY. *Suppose that $f \in S^k(\Gamma_0(N))$ is a Hecke eigenform (newform) of weight $k \equiv 2 \bmod 4$ such that $L(f, s)$ does not vanish at $s = k/2$. Then there exists a Hecke eigenform (newform)*

$$g \in S_{(k/2)+1}(\Gamma_0(N)) \qquad \textit{satisfying} \quad L(f \otimes \operatorname{Sym}^2(g), k) \neq 0$$

or there exists a Hecke eigenform (newform)

$$g \in S_{(k/2)+3}(\Gamma_0(N)) \qquad \textit{satisfying} \quad L(f \otimes \operatorname{Sym}^2(g), k + 2) \neq 0.$$

We think it may be quite hard to prove such a statement by other (e.g., analytic) methods. Note that for the full modular group the corollary does not make sense because the triple L-function always has a $-$sign in its functional equation and that for k divisible by 4 there is no g as required.

Proofs. To prove our theorem we will have to analyze the action of certain differential operators on a Jacobi form $F(\tau, z)$ associated to f and constructed with the help of Yoshida's lifting.

As in [5] we denote for $Z=\left(\begin{smallmatrix}\tau & z\\ z & \tau'\end{smallmatrix}\right)$ in the Siegel upper half plane $\mathbf{H}_2$ by

$$Y^{(2)}(1, \varphi)(Z)=\sum_{i,\,j=1}^{r}\frac{1}{e_i e_j}\sum_{\mathbf{x}=(x_1,\,x_2)\in(y_i R y_j^{-1})^2}\varphi(y_j)\times(Im(\overline{x_1}\,x_2))\exp(2\pi i\,\mathrm{tr}(q(\mathbf{x})\,Z)) \tag{3}$$

the second Yoshida lifting of the pair $(1, \varphi)$, it is a Siegel cusp form of degree 2 and weight $2+v$. By results of Yoshida [13] it is zero if and only if its first Fourier–Jacobi coefficient

$$F(\tau, z)=\sum_{j=1}^{r}\frac{1}{e_j}\sum_{x\in(y_j R y_j^{-1})}\varphi(y_j)(Im(x))\exp(2\pi i(q(x)\,\tau+\mathrm{tr}(x)\,z)) \tag{4}$$

is zero. Notice here that the lattices $y_i R y_j^{-1}$ with $i\neq j$ do not contain vectors of norm 1 and therefore do not appear in the first Fourier–Jacobi coefficient. Moreover, the jth summand in this first Fourier–Jacobi coefficient arises as the sum of the Jacobi theta series with respect to the units of the order $R_j=y_j R y_j^{-1}$ which are easily seen to be all the same (shift the summation over x to one over εx). Hence the jth summand is e_j times the Jacobi theta series of R_j with respect to 1.

The choice of decomposition $N=N_1 N_2$ made in the statement of the theorem implies by Theorem 5.1 of [5] that $Y^{(2)}(1, \varphi)(Z)=0$ (or equivalently $F(\tau, z)=0$) if and only if $L(f, k/2)=0$ for the modular form f of weight $k=2+2v$ corresponding to the given φ of weight v. In particular $Y^{(2)}(1, \varphi)=0$ for all ϕ if v is odd since the sign of the functional equation of $L(f, s)$ is $(-1)^{v+1}$ times the eigenvalue of f under w_N.

For $r, k\in\mathbf{N}$ there is a (non-zero) polynomial $p_{r,k}(x_1, x_2, x_3)$ and a corresponding differential operator

$$\mathscr{D}_{r,k}:=p_{r,k}\left(\frac{\partial}{\partial\tau}, \frac{\partial}{\partial z}, \frac{\partial}{\partial\tau'}\right)_{|z=0}$$

mapping functions F defined on $\mathbf{H_2}$ to functions on $\mathbf{H}\times\mathbf{H}$ such that the equivariance properties are satisfied for all $M=\left(\begin{smallmatrix}a & c\\ b & d\end{smallmatrix}\right)\in SL_2(\mathbf{R})$ as

$$\mathscr{D}_{r,k}(F|_k M^{\uparrow})=(\mathscr{D}_{r,k}F)|_{k+r}^{\tau} M,$$

where $\uparrow$ denotes the embedding of $SL_2(\mathbf{R})$ into $Sp(2, \mathbf{R})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the upper index τ indicates that the operator $|_{k+r} M$ should be applied with respect to the variable τ. The polynomial $p_{k,r}$ is essentially a Gegenbauer (or ultraspherical) polynomial (see [7, 3]), we can write down the differential operator $\mathscr{D}_{r,k}$ quite explicitly (after choosing some normalization) as

$$\mathscr{D}_{r,k} = \sum_{\mu=0}^{[r/2]} (-1)^{\mu} \frac{(k+r-\mu-2)!}{\mu!\,(r-2\mu)!}\,(k+r-\mu-2)! \left(\frac{\partial}{\partial\tau}\,\frac{\partial}{\partial\tau'}\right)^{\mu} \left(\frac{\partial}{\partial z}\right)^{r-2\mu}.$$

It is well-known that for any fixed $\mathfrak{Y} \in \mathbf{C}^{2k}$, the function

$$\begin{cases} \mathbf{C}^{2k} \to \mathbf{C} \\ \mathfrak{X} \mapsto p_{r,k}(\mathfrak{X}^t\mathfrak{X},\, 2\mathfrak{X}^t\mathfrak{Y},\, \mathfrak{Y}^t\mathfrak{Y}) \end{cases}$$

is then a harmonic form of weight r in $m = 2k$ variables, see [2, 7]. We need here a more general property of these polynomials.

Proposition. *Let $P\colon \mathbf{C}^{(m,2)} = M_{m,2}(\mathbf{C}) \to \mathbf{C}$ be a polynomial function which as a function of the first column is harmonic and satisfies*

$$P\left((\mathfrak{X}, \mathfrak{Y}) \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\right) = a^{\mu} P(\mathfrak{X}, \mathfrak{Y})$$

for all $a \in \mathbf{C}^{\star}$, $b \in \mathbf{C}$.

Then, for any fixed $\mathfrak{Y} \in \mathbf{C}^m$, the function

$$\begin{cases} \mathbf{C}^{m} \to \mathbf{C} \\ \mathfrak{X} \mapsto p_{r,(m/2)+\mu}(\mathfrak{X}^t\mathfrak{X},\, 2\mathfrak{X}^t\mathfrak{Y},\, \mathfrak{Y}^t\mathfrak{Y})\, P(\mathfrak{X}, \mathfrak{Y}) \end{cases}$$

is a homogeneous harmonic polynomial of degree $r+\mu$.

Proof. We denote the function in question by $\mathfrak{X} \mapsto R_{\mathfrak{Y}}(\mathfrak{X}) = R^{(r)}_{\mathfrak{Y},P}(\mathfrak{X})$ and consider for $\tau = u + iv \in \mathbf{H}$ and $w \in \mathbf{C}^m$ the function

$$f(\tau, w) = \sum_{\mathfrak{X} \in \mathbf{Z}^m} R_{\mathfrak{Y}}(\mathfrak{X} + w)\, e^{\pi i (\mathfrak{X}+w)^t (\mathfrak{X}+w)\tau}.$$

This is a periodic function of w, therefore it has a Fourier expansion of the form

$$f(\tau, w) = \sum_{G \in \mathbf{Z}^m} A(G, \tau)\, e^{2\pi i G^t w}.$$

We compute the Fourier coefficients in two ways: First of all as in [2] one gets

$$A(G, iv) = v^{-(m-\mu-r)/2} e^{-\pi v^{-1} G^t G} \hat{R}_{\mathfrak{Y}}(-iGv^{-1/2}),$$

where $\hat{R}_{\mathfrak{Y}}$ denotes the Gauß transform of the polynomial $R_{\mathfrak{Y}}$. We recall that the Gauß transform $\hat{Q}$ of a polynomial function $Q\colon \mathbf{C}^m \to \mathbf{C}$ is defined by

$$\hat{Q}(\mathfrak{X}) = \int_{\mathbf{R}^m} Q(\mathfrak{X} + \mathfrak{y}) \exp(-\pi\, \mathfrak{y}^t \cdot \mathfrak{y})\, d\mathfrak{y}.$$

On the other hand we start (writing just $P(\mathfrak{X})$ instead of $P(\mathfrak{X}, \mathfrak{Y})$) from the function

$$\begin{aligned} g(\tau, z, \tau', w) &= \sum_{\mathfrak{X} \in \mathbf{Z}^m} P(\mathfrak{X} + w)\, e^{\pi i \{(\mathfrak{X}+w)^t (\mathfrak{X}+w)\tau + 2(\mathfrak{X}+w)^t \mathfrak{Y} z + \mathfrak{Y}^t \mathfrak{Y} \tau'\}} \\ &= e^{\pi i \mathfrak{Y}^t \mathfrak{Y} \tau'} \vartheta_P \begin{bmatrix} w \\ \mathfrak{Y} z \end{bmatrix} (1_m, \tau), \end{aligned}$$

where (as in [8]) the theta constant $\vartheta_P[{A \atop B}]$ is (for $A, B \in \mathbf{C}^m$) defined by

$$\vartheta_P \begin{bmatrix} A \\ B \end{bmatrix} (1_m, \tau) = \sum_{\mathfrak{X} \in \mathbf{Z}^m} P(\mathfrak{X} + A)\, e^{\pi i \{(\mathfrak{X}+A)^t (\mathfrak{X}+A)\tau + 2B^t(\mathfrak{X}+A)\}}.$$

The reciprocity law for such series (see [8, p. 53]) states that

$$\vartheta_P \begin{bmatrix} A \\ B \end{bmatrix} \left(1_m, -\frac{1}{\tau}\right) = e^{2\pi i A^t B} \left(\frac{\tau}{i}\right)^{m/2} \vartheta_Q \begin{bmatrix} -B \\ \mathbf{A} \end{bmatrix} (1_m, \tau),$$

with $Q(\mathfrak{X}) = (\tau/i)^{\mu}\, (-i)^{\mu}\, P(\mathfrak{X})$. After a standard calculation and taking into account the special property $P(\mathfrak{X} - \mathfrak{Y} z) = P(\mathfrak{X})$ we obtain from this

$$\begin{aligned} &g(\tau, z, \tau', w) \\ &\quad = i^{m/2} \left(\sum_{\mathfrak{X} \in \mathbf{Z}^m} P(\mathfrak{X})\, e^{\pi i \{\operatorname{trace}((\mathfrak{X}, \mathfrak{Y})^t (\mathfrak{X}, \mathfrak{Y}) \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) + 2w^t \mathfrak{X}\}} \right) \Bigg|_{(m/2)+\mu} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\uparrow}. \end{aligned}$$

Applying the differential operator $\mathscr{D}_{r,k+\mu}$ and using its equivariance properties we obtain from the equation above

$$e^{\pi i \mathfrak{Y}^t \mathfrak{Y} \tau'} f(\tau, w) = e^{\pi i \mathfrak{Y}^t \mathfrak{Y} \tau'} i^{m/2} (\tau)^{-(m/2)-\mu-r} \sum_{\mathfrak{X} \in \mathbf{Z}^m} R_{\mathfrak{Y}}(\mathfrak{X})\, e^{\pi i \{\mathfrak{X}^t \mathfrak{X}(-1/\tau) + 2\mathfrak{X}^t w\}}. \tag{5}$$

Comparing with our first formula for the Fourier coefficients gives for all $G \in \mathbf{Z}^{\mathbf{m}}$:

$$\hat{R}_{\mathfrak{Y}}(-iG) = R_{\mathfrak{Y}}(-iG).$$

This implies $\hat{R}_{\mathfrak{Y}} = R_{\mathfrak{Y}}$, hence $R_{\mathfrak{Y}}$ is indeed harmonic ([8]).

As in [4, Section 5] we use the polynomial $p_{2v,2}$ from above to define (for $1 \leqslant i, j \leqslant h$) the U_{2v}-valued theta series

$$\tilde{\Theta}_j^{2v}(\tau, x) = \sum_{x' \in (y_j R y_j^{-1})} p_{2v,2}(n(x), \operatorname{tr}(x\overline{x'}), (n(x')) \exp(2\pi i n(x) \tau). \tag{6}$$

Let

$$\theta_j(P, \tau, z) = \sum_{x \in (y_j R y_j^{-1})} P(Im(x)) \exp(2\pi i (q(x) \tau + \operatorname{tr}(x) z))$$

for $P \in U_\mu^{(0)}$ and

$$\theta_j(Q, \tau) = \sum_{x \in (y_j R y_j^{-1})} Q(x) \exp(2\pi i (q(x) \tau)$$

for $Q \in U_{2v}$.

We have then the following

Lemma. *For all polynomials $P \in U_\mu^{(0)}$ and $v \in \mathbf{N}$ with $2v \geqslant \mu$ one has*

$$T(P, \tilde{\Theta}_j^{2v}(\tau, \cdot)) \exp(2\pi i \tau') = c\mathscr{D}_{2v-\mu, 2+\mu}(\theta_j(P, \tau, z) \exp(2\pi i \tau')) \tag{7}$$

with some constant $c \neq 0$.

Proof. For $P \in U_\mu^{(0)}$ the polynomial function $\tilde{P}: (x, x') \mapsto P(Im(x\overline{x'}))$ (interpreted as a function of the coordinates of x, x' with respect to the fixed orthonormal basis of D_∞) satisfies the conditions of the proposition. Applying the differential operator $\mathscr{D}_{2v-\mu, 2+\mu}$ to $\theta_j(P, \tau, z) \exp(2\pi i \tau')$ for $P \in U_\mu^{(0)}$ we obtain (by explicitly computing the action of the differential operator as in formula (5) and writing $\mathbf{1}$ for the identity element of $D^\times$)

$$\mathscr{D}_{2v-\mu, 2+\mu}(\theta_j(P, \tau, z) \exp(2\pi i \tau')) = \theta_j(P_{2v}(P), \tau) \exp(2\pi i \tau') \tag{8}$$

with

$$(P_{2\nu}(P))(x) := R_{\mathbf{1},P}^{(2\nu-\mu)}(x) = p_{2\nu-\mu,\,2+\mu}(n(x), \mathrm{tr}(x), n(\mathbf{1}))\,\tilde{P}(x, \mathbf{1}), \tag{9}$$

which is harmonic by the proposition. The (nonzero) map $P \mapsto P_{2\nu}(P)$ from $U_\mu^{(0)}$ into $U_{2\nu}$ is intertwining with respect to the action of $D_\infty^\times$ on these spaces. The same is true for the map defined by associating to $P \in U_\mu^{(0)}$ the polynomial $\mathfrak{Y} \mapsto T(P, R_{\mathfrak{Y},1}^{(2\nu)})$ with $R_{\mathfrak{Y},1}^{(2\nu)}(\mathfrak{X}) = p_{2\nu,2}(n(x), \mathrm{tr}(x\overline{x'}), n(x'))$ (where $\mathfrak{X}, \mathfrak{Y}$ are the coordinate vectors with respect to an orthonormal basis of D_∞ of x, x' respectively) as in the proof of the proposition. This follows from the well known invariance properties of the reproducing kernel function $R_{\mathfrak{Y},1}^{(2\nu)}(\mathfrak{X})$ for $U_{2\nu} = U_\nu^{(0)} \otimes U_\nu^{(0)}$ and the invariance of T. By the uniqueness of an intertwining map between these spaces the two maps must be proportional, which proves the lemma.

Proof of Theorem 1. From the remarks made before the statement of the Theorem we know that $L(f, k/2) = 0$ is equivalent to the vanishing of $F(\tau, z)$. As in [7, Section 3], [6] one sees by considering the Taylor expansion with respect to z of the Jacobi form $F(\tau, z)$ of index 1 that F is identically zero if and only if all the functions $F_r(\tau)$ given by $F_r(\tau)\exp(2\pi i\tau') = \mathscr{D}_{r,2+\mu}(F(\tau, z)\exp(2\pi i\tau'))$ are zero. This is again equivalent to $F(\tau, 0) = F_2(\tau) = 0$. We note that by (8)

$$F_r(\tau) = \sum_{j=1}^{h} \frac{1}{e_j}\,\theta_j(P_{2\nu}(\varphi(y_j)), \tau). \tag{10}$$

Let f be an elliptic cusp form of square free level N. Then f is zero if and only if for all newforms g of the same weight and of level M dividing N the Petersson products $\langle f, g \mid \prod_{p\in S'} w_p\rangle$ are zero for all sets S' consisting of primes p dividing N/M. We see therefore that $F(\tau, z) = 0$ for all τ, z is equivalent to the vanishing of all the Petersson products $\langle F_r, g \mid \prod_{p\in S'} w_p\rangle$ for all newforms g of weight $2 + 2r + \mu$ $(r = 0, 2)$ and level M dividing N and all subsets S' of the set S of primes p dividing N/M.

To evaluate these Petersson products we recall from [4, Lemma 5.3] that for a newform g of weight $2 + 2\nu$ and level M dividing N one has

$$\langle g, \tilde{\Theta}_j^{(2\nu)}(\tau, 0)\rangle = \langle g, g\rangle \sum_{S''\subseteq S} \left(\prod_{p\in S''} (\widetilde{w}_p)\,\psi\right)(y_j) \otimes \left(\prod_{p\in S''} (\widetilde{w}_p)\,\psi\right)(y_j), \tag{11}$$

where S is the set of primes dividing N/M, R' is the Eichler order of level M containing R and $\psi \in \mathscr{A}_{\mathrm{ess}}(D_{\mathbf{A}}^\times, (R'_{\mathbf{A}})^\times, \tau_\nu)$ is the essential form corresponding to g under Eichler's correspondence. For $g \mid \prod_{p\in S'} w_p$ the same equation is true with ψ replaced by $\prod_{p \in S'} (\widetilde{w_p})\,\psi$. This is of the same shape as (11) with S'' replaced by the symmetric difference of S', S''.

From the Lemma and (8) we have

$$T(\varphi(y_j), \langle \tilde{\Theta}_j^{2\nu}(\tau, \cdot), g\rangle) = \langle c(\theta_j(P_{2\nu}(\varphi(y_j)), \tau)), g\rangle \tag{12}$$

and hence

$$\langle F_r, g\rangle = c_1 \sum_{j=1}^{h} \frac{1}{e_j} T\left(\varphi(y_j), \sum_{S' \subseteq S} \left(\prod_{p \in S'} (\widetilde{w_p})\,\psi\right)(y_j) \otimes \left(\prod_{p \in S'} (\widetilde{w_p})\,\psi\right)(y_j)\right), \tag{13}$$

and similarly for the equations involving Atkin-Lehner involutions. By assumption φ is orthogonal to all functions invariant under the adelic units of an order in D strictly containing R, hence all the summands on the right hand side here are zero unless $M = N$, i. e., the contributions from oldforms (or non-essential forms on the quaternion side) all vanish automatically.

In the remaining case $M = N$ we obtain

$$\langle F_r, g\rangle = c_1 \sum_{j=1}^{h} \frac{1}{e_j} T(\varphi(y_j), \psi(y_j) \otimes \psi(y_j)) \tag{14}$$

with some nonzero constant c_1. This proves the assertion.

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